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Non-equilibrium critical relaxation in dilute Ising systems

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Abstract. We investigate the critical relaxation of dilute Ising systems starting from a macroscopically prepared initial state with short-range correlations. Using the methods of renormalized field theory we calculate the exponent θ' which describes the initial *increase* of the magnetization to second order in $\sqrt{\epsilon}$, where $\epsilon = 4 - d$. Since computer simulations of the dilute Ising model have shown that a large part of the critical region is governed by crossover phenomena, we also discuss the influence of the slow crossover on the relaxation.

1. Introduction

In recent years the growth of correlations which governs the initial stage of the critical relaxation has been studied for a number of dynamic universality classes [1, 2]. Of special interest are thermodynamic systems with a non-conserved order parameter whose critical dynamics may be described by model A or model C in the terminology of Halperin *et al* [3]. When such a system is quenched from a high temperature $T_0 \gg T_c$ to the critical point, the relaxation displays universal scaling behaviour which is characterized by a new critical exponent θ' already at (macroscopically) short times.

A remarkable property of the relaxation process is the increase of a non-zero initial magnetization M_0 at short times $t < t_M$, where $t_M \sim M_0^{-1/(\theta'+\beta/(\nu z))}$. The crossover from the initial rise to the well known decay $M(t) \sim t^{-\beta/(\nu z)}$ for $t \to \infty$ can be described by the scaling form

$$M(t) = M_0 t^{\theta'} F(t^{\theta'+\beta/(\nu z)} M_0) \tag{1}$$

where

$$F(x) \sim \begin{cases} 1 & \text{for } x \to 0 \\ 1/x & \text{for } x \to \infty\,. \end{cases} \tag{2}$$

For the (pure) Ising model the scaling function F has been calculated by one of us to first order in $\epsilon = 4 - d$ [4], where d is the spatial dimension. A detailed discussion of finite size effects and the relaxation away from criticality is given in [5].

For Ising systems with model-A dynamics the predictions obtained by renormalization group calculations have been successfully checked in simulations [6–8]. Owing to the small correlation length at the beginning of the relaxation these simulations require less effort than numerical studies of equilibrium dynamics. This fact has also been used to develop new methods to measure both dynamic and static critical exponents [9].

In this paper we apply field theory and the renormalization group to study the relaxation of dilute Ising systems with a non-conserved order parameter (model A). As a peculiarity

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of the dilute Ising model the renormalization group possesses no fixed point of the order ϵ. We therefore follow an idea by Khmelnitskii [10] and perform a $\sqrt{\epsilon}$-expansion to calculate the exponent θ'. It has already been shown by Kissner that θ' vanishes at first order in $\sqrt{\epsilon}$ [11]. In section 3 we extend the calculation to two-loop order and obtain the non-trivial result $\theta' = 0.0868\epsilon + O(\epsilon^{3/2})$.

Extensive computer simulations of the dilute Ising model give evidence for a slow crossover to universal criticality [12] (see also [13]). Therefore, asymptotic scaling behaviour with universal critical exponents in general occurs only in the limit of extremely large length and time scales. In section 4 we discuss the influence of crossover phenomena on the critical relaxation. The pre-asymptotic behaviour observed in simulations is characterized by effective exponents which depend on the concentration p of magnetic sites. In a recent publication we have shown how the non-universal exponents are related to regions in the space of coupling constants away from the fixed point [14]. Here we apply this method to estimate non-universal values of θ' which may be measured in future simulations.

2. The model

The dynamics of a non-conserved order parameter field s in a system with quenched random impurities can be expressed in the form of the Langevin-equation

$$\partial_t s(r,t) = -\lambda \frac{\delta \mathcal{H}_\psi[s]}{\delta s(r,t)} + \zeta(r,t) \tag{3}$$

with the Landau–Ginzburg Hamiltonian

$$\mathcal{H}_\psi[s] = \int \mathrm{d}^d r \left[\tfrac{1}{2}(\tau + \psi)s^2 + \tfrac{1}{2}(\nabla s)^2 + \frac{1}{4!} g s^4 \right]. \tag{4}$$

Here ψ is a quenched (time-independent) Gaussian random field with zero mean and the correlations

$$\overline{\psi(r)\psi(r')} = f\delta(r - r'). \tag{5}$$

The bar denotes the average over disorder. The Gaussian random force ζ in (3) models the effect of microscopic degrees of freedom with short relaxation times:

$$\langle \zeta(r,t)\zeta(r',t') \rangle = 2\lambda\delta(r - r')\delta(t - t'). \tag{6}$$

The angular brackets $\langle \cdots \rangle$ indicate an average with respect to thermal noise.

In this paper we consider the case of a one-component order parameter (universality class of the Ising model) since the specific heat exponent α of the pure Ising model is positive for $d = 3$. For systems with negative α (like the three-dimensional n-vector model for $n \geqslant 2$) random impurities are irrelevant for the asymptotic critical behaviour [15].

A field theoretic formulation of the dynamics which is equivalent to the Langevin equation (3) is defined by the stochastic functional [4, 16, 17, 18]

$$\mathcal{J}_\psi[\tilde{s}, s] = \int_0^\infty \mathrm{d}t \int \mathrm{d}^d r\, \tilde{s} \left[\partial_t s + \lambda \frac{\delta \mathcal{H}_\psi[s]}{\delta s} - \lambda \tilde{s} \right] \tag{7}$$

where the response field $\tilde{s}$ has been introduced to average over thermal noise.

The Langevin equation (3) and the dynamic functional (7) allow us to investigate the equilibrium critical dynamics of disordered Ising systems. Since we are interested in the relaxation from a macroscopically prepared non-equilibrium initial state we additionally

have to specify the distribution of the initial condition $s_0(r) = s(r, t = 0)$. A quench from a high temperature $T_0 \gg T_c$ at the time $t = 0$ corresponds to the distribution

$$\mathcal{P}_0[s] \propto \exp\left[-\frac{\tau_0}{2}\int \mathrm{d}^d r\ (s_0(r) - M_0)^2\right] \tag{8}$$

where M_0 is a homogeneous initial magnetization and τ_0^{-1} measures the width of the initial distribution. By naive dimensional analysis one finds $\tau_0 \sim \mu^2$ (μ is an external momentum scale). Thus τ_0^{-1} is an irrelevant parameter in the renormalization group sense. The fixed point $\tau_0 = \infty$ corresponds to a sharp preparation of the initial value $s_0(r) = M_0$.

Correlation and response functions are computed by functional integrals of the form

$$\overline{\langle s(r,t)\ldots\rangle} = \int \mathcal{D}[\psi;\tilde{s},s;s_0]\, s(r,t)\ldots\exp(-\mathcal{J}_\psi[\tilde{s},s])\mathcal{P}_0[s_0]\ . \tag{9}$$

We perform the average over disorder at the beginning of the calculations [19]

$$\int \mathcal{D}[\psi]\ \exp(-\mathcal{J}_\psi[\tilde{s},s]) = \overline{\exp(-\mathcal{J}_\psi[\tilde{s},s])} = \exp(-\mathcal{J}[\tilde{s},s]) \tag{10}$$

and obtain the ψ-independent stochastic functional

$$\mathcal{J}[\tilde{s},s] = \int \mathrm{d}^d r\left[\int_0^\infty \mathrm{d}t\,\tilde{s}\left(\partial_t s + \lambda(\tau - \Delta)s + \frac{\lambda g}{6}s^3 - \lambda\tilde{s}\right) - \tfrac{1}{2}\lambda^2 f\left(\int_0^\infty \mathrm{d}t\,\tilde{s}s\right)^2\right]. \tag{11}$$

In the next section we will study the Green functions of this field theory by an expansion around the Gaussian model ($f = g = 0$). The propagator and the correlator of the 'free' field theory are

$$\begin{aligned} G_q(t-t') &= \int \mathrm{d}^d r\ \exp(-\mathrm{i}q\cdot r)\langle s(r,t)\tilde{s}(\mathbf{0},t')\rangle^{(0)} \\ &= \exp(-\lambda(\tau+q^2)(t-t')) \qquad \text{for } t > t' \end{aligned} \tag{12}$$

and

$$C_q(t,t') = \int \mathrm{d}^d r\ \langle s(r,t)s(r',t')\rangle^{(0)} = C_q^{(\mathrm{eq})}(t-t') + C_q^{(\mathrm{i})}(t,t') \tag{13}$$

respectively. Here

$$C_q^{(\mathrm{eq})}(t-t') = \frac{1}{\tau+q^2}\exp\left(-\lambda(\tau+q^2)|t-t'|\right) \tag{14}$$

is the equilibrium correlator and

$$C_q^{(\mathrm{i})}(t,t') = \left(\tau_0^{-1} - \frac{1}{\tau+q^2}\right)\exp\left(-\lambda(\tau+q^2)(t+t')\right) \tag{15}$$

is a non-equilibrium contribution which results from the initial conditions.

3. Renormalization group analysis

In this section we use the methods of renormalized field theory to investigate the scaling behaviour of non-equilibrium response and correlation functions. For a more detailed exposition of the procedure the reader is referred to [4].

The Green functions $G^{\tilde{L}}_{\tilde{N},N}$ are defined as cumulants of $(\tilde{L}+\tilde{N})$ response fields ($\tilde{L}$ of them at time $t=0$) and N order parameter fields,

$$G^{\tilde{L}}_{\tilde{N},N}(\{x,\tilde{r},\tilde{t},r,t\}) = \left\langle \prod_{i=1}^{\tilde{L}} \tilde{s}_0(x_i) \cdot \prod_{j=1}^{\tilde{N}} \tilde{s}(\tilde{r}_j,\tilde{t}_j) \cdot \prod_{k=1}^{N} s(r_k,t_k) \right\rangle . \tag{16}$$

(Here and in what follows the brackets are meant to include the average over disorder.) For $\tau_0=\infty$ and $M_0=0$ the initial order parameter field is sharply defined as $s_0(r)=0$. Insertions of the time derivative $\dot{s}_0(r)=\partial_t s(r,t)|_{t=0}$ in Green functions are related to the response field $\tilde{s}_0(r)$ by [4]

$$\dot{s}_0(r) = 2\lambda\tilde{s}_0(r) . \tag{17}$$

For finite τ_0 insertions of the field $s_0(r)$ are non-zero, and we have

$$s_0(r) = \tau_0^{-1}\tilde{s}_0(r) . \tag{18}$$

The functions $G^{\tilde{L}}_{\tilde{N},N}$ may be calculated by a perturbation expansion in the coupling coefficients f and g. We use dimensional regularization to calculate otherwise divergent integrals and absorb the remaining poles in ϵ into reparametrizations of coupling coefficients and fields. The required renormalizations are

$$\begin{aligned} &\tilde{s} \to \mathring{\tilde{s}} = Z_{\tilde{s}}^{1/2}\tilde{s} && s \to \mathring{s} = Z_s^{1/2}s \\ &\tau \to \mathring{\tau} = (Z_\tau/Z_s)\tau && \lambda \to \mathring{\lambda} = (Z_s/Z_{\tilde{s}})^{1/2}\lambda \\ &g \to \mathring{g} = S_d^{-1}(Z_u/Z_s^2)u\mu^\epsilon && f \to \mathring{f} = S_d^{-1}(Z_v/Z_s^2)v\mu^\epsilon \end{aligned} \tag{19}$$

where μ is an external momentum scale and S_d denotes the surface area of the d-dimensional unit sphere divided by $(2\pi)^d$. The Z-factors in (19) have been calculated by Lawrie and Prudnikov to two-loop order [20] (see also [14]).

Since the non-equilibrium initial conditions break the translational invariance with respect to time the initial response field requires an additional renormalization

$$\tilde{s}_0 \to \mathring{\tilde{s}}_0 = (Z_0 Z_{\tilde{s}})^{1/2}\tilde{s}_0 . \tag{20}$$

The new Z-factor Z_0 serves to cancel the divergencies in response functions with insertions of the field $\tilde{s}_0$, e.g.

$$(Z_0 Z_{\tilde{s}} Z_s)^{-1/2}\mathring{G}^{1}_{0,1}(r,t) = \text{finite}.$$

To two-loop order we find

$$Z_0 = 1 + \frac{u}{2\epsilon} + \frac{u^2}{4\epsilon^2}\left[2 + \left(\ln 2 - \tfrac{1}{2}\right)\epsilon\right] + \frac{uv}{4\epsilon^2}\left[-6 + \left(2 + \ln 2 - \sqrt{3}\ln(2+\sqrt{3})\right)\epsilon\right] . \tag{21}$$

Since the bare Green functions $\mathring{G}^{\tilde{L}}_{\tilde{N},N}$ are independent of the momentum scale μ introduced in (19), the μ-derivatives at fixed bare parameters are zero:

$$0 = \mu\left.\frac{\mathrm{d}}{\mathrm{d}\mu}\right|_0 \mathring{G}^{\tilde{L}}_{\tilde{N},N} = \mu\left.\frac{\mathrm{d}}{\mathrm{d}\mu}\right|_0 (Z_0 Z_{\tilde{s}})^{\tilde{L}/2} Z_{\tilde{s}}^{\tilde{N}/2} Z_s^{N/2} G^{\tilde{L}}_{\tilde{N},N} . \tag{22}$$

Thus (for $M_0=0$) the renormalized Green functions satisfy the renormalization group equation (RGE)

$$\begin{aligned} &\Big[\mu\partial_\mu + \kappa_\lambda\lambda\partial_\lambda + \kappa_\tau\tau\partial_\tau + \beta_u\partial_u + \beta_v\partial_v + \tfrac{1}{2}\tilde{L}(\gamma_0+\gamma_{\tilde{s}}) \\ &\quad + \tfrac{1}{2}\tilde{N}\gamma_{\tilde{s}} + \tfrac{1}{2}N\gamma_s\Big] G^{\tilde{L}}_{\tilde{N},N}(\{r,t\};\tau,u,v,\lambda;\mu) = 0 \end{aligned} \tag{23}$$

with the Callan–Symanzik functions $\beta_u = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\big|_0 u$ and $\beta_v = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\big|_0 v$ and the Wilson functions

$$\begin{aligned} \gamma_0 &= \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln Z_0 \qquad \gamma_{\tilde{s}} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln Z_{\tilde{s}} \qquad \gamma_s = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln Z_s \\ \kappa_\lambda &= \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln \lambda = (\gamma_{\tilde{s}} - \gamma_s)/2 \qquad \kappa_\tau = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_0 \ln \tau. \end{aligned} \tag{24}$$

Using equation (21) the new Wilson function γ_0 can be calculated to two-loop order

$$\gamma_0(u, v) = -\tfrac{1}{2}u - \tfrac{1}{4}u^2(2\ln 2 - 1) - \tfrac{1}{2}uv\left(2 + \ln 2 - \sqrt{3}\ln(2+\sqrt{3})\right). \tag{25}$$

For the discussion of the decay of the initial magnetization M_0 we will also require κ_λ which is given by

$$\kappa_\lambda(u, v) = v + \tfrac{1}{24}(6\ln\tfrac{4}{3} - 1)u^2 - \tfrac{1}{4}uv + \tfrac{5}{4}v^2. \tag{26}$$

The other Wilson functions have been obtained in the minimal renormalization scheme to three-loop order [14] and directly in $d = 3$ to four-loop order [21].

The general solution of the RGE may be written in the form

$$\begin{aligned} G^{\tilde{L}}_{\tilde{N},N}(\{\boldsymbol{x}, r, t\}; \tau, u, v, \lambda; \mu) &= X_0(l)^{\tilde{L}/2}(X_{\tilde{s}}(l)l^{d+2})^{(\tilde{L}+\tilde{N})/2}(X_s(l)l^{d-2})^{N/2} \\ &\times G^{\tilde{L}}_{\tilde{N},N}(\{l\boldsymbol{x}, lr, Y_\lambda(l)l^2 t\}; Y_\tau(l)l^{-2}\tau, \bar{u}(l), \bar{v}(l), \lambda; \mu) \end{aligned} \tag{27}$$

where the characteristics $X_a(l)$ and $Y_b(l)$ are solutions of the ordinary differential equations

$$\begin{aligned} l\frac{d}{dl}\ln X_a(l) &= \gamma_a(\bar{u}(l), \bar{v}(l)) \qquad l\frac{d}{dl}\ln Y_b(l) = \kappa_b(\bar{u}(l), \bar{v}(l)) \\ l\frac{d}{dl}\bar{u}(l) &= \beta_u(\bar{u}(l), \bar{v}(l)) \qquad l\frac{d}{dl}\bar{v}(l) = \beta_v(\bar{u}(l), \bar{v}(l)) \\ X_a(1) &= Y_b(1) = 1 \qquad \bar{u}(1) = u \qquad \bar{v}(1) = v \end{aligned} \tag{28}$$

where $a = 0, \tilde{s}, s$ and $b = \lambda, \tau$.

Equation (27) allows us to study the scaling behaviour of Green functions on large length and time scales. For this purpose one is interested in the limit $l \to 0$ which is governed by the infra-red-stable fixed point

$$\begin{aligned} u_* &= 4\sqrt{\frac{6\epsilon}{53}} - \frac{72}{53^2}(19 + 21\zeta(3))\epsilon + \mathrm{O}(\epsilon^{3/2}) \\ v_* &= \sqrt{\frac{6\epsilon}{53}} - \frac{6}{53^2}(110 + 63\zeta(3))\epsilon + \mathrm{O}(\epsilon^{3/2}) \end{aligned} \tag{29}$$

(where ζ denotes Riemann's ζ-function). The critical exponents η_0, $\tilde{\eta}$, η, ν, and z are the fixed-point values of the Wilson functions γ_0, $\gamma_{\tilde{s}}$, γ_s, $1/(2-\kappa_\tau)$, and $2+\kappa_\lambda$, respectively, and the characteristics show the limiting behaviour

$$\begin{aligned} X_0(l) &\simeq X_0^* l^{\eta_0} \qquad X_{\tilde{s}} \simeq X_{\tilde{s}}^* l^{\tilde{\eta}} \qquad X_s \simeq X_s^* l^{\eta} \\ Y_\tau(l) &\simeq Y_\tau^* l^{2-1/\nu} \qquad Y_\lambda(l) \simeq Y_\lambda^* l^{z-2} \end{aligned} \tag{30}$$

for $l \to 0$ with the non-universal scaling factors X_0^*, $X_{\tilde{s}}^*$, X_s^*, Y_τ^*, and $Y_\lambda^* = (X_{\tilde{s}}^*/X_s^*)^{1/2}$. The new exponent η_0 and the dynamic scaling exponent z are given by

$$\begin{aligned} \eta_0 &= -2\sqrt{\tfrac{6}{53}}\epsilon + \left(\tfrac{57}{53} + \tfrac{63}{53}\zeta(3) - 5\ln 2 + \sqrt{3}\ln(2+\sqrt{3})\right)\tfrac{12}{53}\epsilon + \mathrm{O}(\epsilon^{3/2}) \\ &= -0.673\sqrt{\epsilon}(1 - 0.444\sqrt{\epsilon}) + \mathrm{O}(\epsilon^{3/2}) \end{aligned} \tag{31}$$

and

$$z = 2 + \sqrt{\tfrac{6}{53}\epsilon} + \left(48 \ln \tfrac{4}{3} - \tfrac{1585}{53} - \tfrac{756}{53}\zeta(3)\right) \tfrac{1}{106}\epsilon + O(\epsilon^{3/2})$$
$$= 2 + 0.336\sqrt{\epsilon}(1 - 0.932\sqrt{\epsilon}) + O(\epsilon^{3/2}) \tag{32}$$

respectively. Setting $\epsilon = 1$ we obtain the estimates $\eta_0 \approx -0.374$ and $z \approx 2.023$. The large coefficient of the $O(\epsilon)$ contribution in equation (32) suggests that the result obtained by the naive $\sqrt{\epsilon}$-expansion probably is not very accurate. In [14] we have improved the three-loop expansion of the β-functions by a Padé–Borel approximation and used the result to calculate the fixed point $(u_*, v_*) = (1.424, 0.144)$. Inserting these values into κ_λ we obtain $z = 2.180$.. A direct two-loop calculation in three dimensions yields $z = 2.237$ [22].

In computer simulations of non-equilibrium critical relaxation one usually studies the decay of an initial magnetization M_0 [7] or autocorrelation and response functions related to $G^1_{0,1}$ [6, 8]. At the fixed point $G^1_{0,1}$ displays the scaling behaviour

$$G^1_{0,1}(r, t; \tau, \lambda; \mu) = r^{-d+z\theta'} F(r^z/t, r\tau^\nu) \tag{33}$$

where

$$\theta' = -(\eta_0 + \tilde{\eta} + \eta)/(2z) \tag{34}$$
$$= \tfrac{3}{106}\epsilon(3 - 6\ln 2 + 8\ln 3 - 2\sqrt{3}\ln(2 + \sqrt{3})) + O(\epsilon^{3/2})$$
$$= 0.0868\epsilon + O(\epsilon^{3/2}). \tag{35}$$

If we allow for a non-zero initial magnetization M_0 the time dependence of the order parameter can be calculated by an expansion in powers of M_0:

$$M(t, M_0; \tau, u, v, \lambda; \mu) = \sum_{N=1}^{\infty} \frac{1}{N!} \int d^d r_1 \cdots \int d^d r_N \; G^N_{0,1}(\{r\}, t; \tau, u, v, \lambda; \mu) M_0^N. \tag{36}$$

Solving the RGE for $M(t)$ yields

$$M(t, M_0; \tau, u, v, \lambda; \mu) = \Big(X(l))^{1/2} l^{(d-2)/2}$$
$$\times M(Y_\lambda(l) l^2 t, (X_{\tilde{s}}(l) X_0(l))^{1/2} l^{-(d-2)/2} M_0; Y_\tau(l) l^{-2}\tau, \bar{u}(l), \bar{v}(l), \lambda; \mu\Big) \tag{37}$$

and the asymptotic scaling form

$$M(t, M_0; \tau, u, v, \lambda; \mu) \simeq M_0 t^{\theta'} F_M(M_0 t^{\theta' + \beta/(\nu z)}, t\tau^{\nu z}) \tag{38}$$

where $F_M(x, y)$ remains finite and non-zero for $x, y \to 0$. For the pure model A the long time behaviour of the magnetization is discussed in reference [5]. We expect the general scaling arguments used therein to apply also to systems with random impurities.

4. Crossover phenomena and effective exponents

Since the asymptotic power laws (30) are only valid for small l the universal scaling behaviour of Green functions is restricted to the limit of large length and time scales. Here the precise meaning of 'large' depends on the system under consideration. In computer simulations of the dilute Ising model the linear size L of the system is an additional length scale which controls the approach to the asymptotic scaling regime. While the pure system already reaches the finite-size scaling limit at small system sizes, the disordered Ising model requires considerably larger values of L to display universal behaviour. Monte

Carlo simulations of systems up to $L = 60$ at various concentrations produced apparently concentration dependent critical exponents [12].

Very recently we have shown how these crossover effects can be explained in the framework of the renormalization group [14]. The numerical solution of the flow equations (28) shows that the scale-dependent coupling constants $(\bar{u}(l), \bar{v}(l))$ first approach a one-dimensional submanifold of the (u, v)-space before they move towards the infra-red-stable fixed point (u_*, v_*). This suggests that the slow crossover observed in the simulations reflects the behaviour of the renormalization group transformation along this 'slow' manifold. In [14] we have used this idea to identify coupling coefficients (u_p, v_p) for each concentration $p \in \{0.6, 0.8, 0.95, 0.9\}$, which reproduce the exponents $\nu_{\text{eff}}(p)$ and $\gamma_{\text{eff}}(p)$ found in the simulations by Heuer [12] with satisfactory accuracy.

We expect that for short times t after the quench the relaxation is affected by crossover phenomena characterized by non-asymptotic values of the exponent θ'. Since the correlation length is small at the beginning of the relaxation finite size effects are negligible up to times of the order $t_L \sim L^z$ [5]. For $t < t_L$ the time scale t controls the approach to the asymptotic scaling limit. To obtain estimates for the dilution-dependent effective exponent in a cubic lattice we have calculated the function

$$\theta'(u, v) = -(\eta_0(u, v) + \tilde{\eta}(u, v) + \eta(u, v))/(2\, z(u, v)) \tag{39}$$

$$= \tfrac{1}{8} u - \tfrac{1}{2} v + \tfrac{1}{16} (2 \ln \tfrac{3}{2} - 1)\, u^2 + \tfrac{1}{16} (7 + 2 \ln 2 - 2\sqrt{3} \ln(2 + \sqrt{3}))\, u\, v - \tfrac{1}{2} v^2 + \text{O(three-loop)} \tag{40}$$

for the coupling coefficients (u_p, v_p) found in [14]. The results given in table 1 show that the effective θ' increases for decreasing concentration p. For the pure system ($p = 1$) our result ($\theta' \approx 0.11$) can be compared with the values $\theta' = 0.102(2)$ [7] and $\theta' = 0.104(3)$ [8] obtained in Monte-Carlo simulations.

Table 1. The concentration-dependent effective exponent θ' calculated with the effective coupling constants obtained in [14]. The infra-red-stable fixed point of the randomly diluted Ising model is labelled by (R). The exponents have been calculated (a) by the two-loop expansion (40), and (b) directly by equation (39) using the two-loop expressions for η_0, $\tilde{\eta}$, η, and z.

p	u_p	v_p	θ' (a)	θ' (b)
(R)	1.424	0.144	0.121	0.118
1.0	0.982	0	0.111	0.110
0.95	1.182	0.066	0.115	0.113
0.9	1.375	0.128	0.119	0.117
0.8	1.647	0.216	0.127	0.122
0.6	2.353	0.444	0.158	0.134

5. Summary

In this paper we have studied the relaxation of dilute Ising systems with a non-conserved order parameter after a quench from a high temperature $T_0 \gg T_c$ to the critical temperature T_c. The relaxation displays the typical short-time scaling behaviour associated with the growth of correlations, which has already been observed in pure systems. To show that

a non-zero initial magnetization increases during the initial stage of the relaxation it was necessary to calculate the exponent θ' to second order in $\sqrt{\epsilon}$.

Due to the slow crossover in dilute Ising systems, it is probably difficult to measure the asymptotic exponent θ' in simulations or real experiments. The pre-asymptotic behaviour can be described by effective coupling coefficients that differ from their fixed-point values. We have used the effective coupling constants calculated in [14] to obtain estimates for dilution-dependent values of the exponent θ' which may be measured in future simulations of the relaxation. Since these values are non-universal quantities they only apply to a particular microscopic realization of the Ising model (they depend, e.g., on the coordination number of the lattice as well as on the dilution) and a limited range of length and time scales. The exponents given in table 1 refer to the cubic lattice with site disorder used by Heuer [12]. For $t \to \infty$ in an infinite system the relaxation is governed by universal exponents for all concentrations above the percolation threshold.

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